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Photocount statistics of gaussian light of arbitrary spectral profile

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Abstract. A general method of deriving the generating function of the photocounts due to gaussian light of arbitrary spectral profile is presented. The eigenvalues of the basic integral equation corresponding to a general real-valued autocorrelation function are related to the usual infinite product representation of the generating function of the photocounts. The integral equation is solved by the Laplace transform technique. The eigenvalues are determined by imposing analyticity requirement on the Laplace transform and the eigenvalues are identified to be the zeros of an alternant. By appropriate parametrization and multiplication by an alternant the eigenvalues are directly related to the zeros of an entire function. By the use of the Hadamard factorization theorem the generating function is identified to be the reciprocal of the entire function evaluated at a chosen point. The method is extended to cover the superposition of two incoherent beams. A further generalization along the same lines leads to the determination of the generating function corresponding to a superposition of beams of arbitrary spectral profile centred about arbitrary frequencies.

1. Introduction

Photocounting analyses are finding increasing applications not only in optical spectroscopy but in other realms like turbulence. The usual method of arriving at the photocount distribution, essentially due to Mandel (1963), consists in observing that the photocounts are governed by a Poisson distribution with parameter $\alpha E(T)$ where α is the photoefficiency of the detector and

$$E(T) = \int_{t}^{t+T} I(t')\,\mathrm{d}t' \tag{1.1}$$

where T is large compared to the coherence time of the incident beam and $I(t)$ is the intensity of the incident beam. The photocount distribution is arrived at by making an ensemble average of the Poisson distribution over E. This method has been successfully employed for deriving the various statistical characteristics of the photocount distribution, (see, for example, Jakeman and Pike 1969 and Perina and Horak 1969). Jakeman and Pike (1969) have presented a table containing exhaustive information regarding the present state of knowledge of photocount statistics. In recent letters, Troup and Lyons (Troup and Lyons 1969, Lyons and Troup 1970) have presented a method of using counting techniques for light beams of arbitrary bandwidth (see also Arecchi 1965 and Arecchi *et al* 1966) by using modulation techniques. So far, the attempts to generalize the distribution to arbitrary time intervals have been confined to gaussian beams with lorentzian spectral profiles. In view of the importance of photocount statistics, it is worthwhile to examine the feasibility of arriving at the photocount distribution of gaussian light beams of arbitrary spectral profile and centre frequencies. This would enable us to make predictions about mixing of beams of different spectral and coherence characteristics. In this paper, we assume that the autocorrelation function of the analytic signal of frequency ω_0 is given by

$$E\{V(t)V^*(t')\} = f(|t-t'|)\exp \mathrm{i}\omega_0(t-t') \tag{1.2}$$

where f is a real valued function defined on the positive axis. Since it is reasonable to assume that $f \to 0$ for large values of its argumment, there is, therefore, a Laplace transform f^* of f. We shall outline the method of arriving at the generating function governing the photocounts. Assuming that $f^*(z) = g(z)/h(z)$ where $g(z)$ and $h(z)$ are polynomials of degree less than n and equal to n respectively, we note that $f(t)$ is of the form

$$f(t) = \sum_{k,m} \alpha_{km} t^k \exp(-z_m t) \qquad z_m \text{ complex} \tag{1.3}$$

so that $f(t)$ is a combination of lorentzian as well as Poisson profiles. It is possible to extend the result to the case when f^* has an infinite number of poles, the sequence of poles having no limit point in any finite part of the plane (see concluding part of § 2). However, the method presented in this paper is not applicable to the case where f^* is an entire function. A simple example is provided by the light passed through a monochromator, in which case the output is always band-limited. It may be worthwhile in this context to examine the applicability of some of the methods of solving similar integral equations encountered in the theory of dams and storage systems (see Srinivasan 1971 and Cohen 1969).

The layout of the paper is as follows. We present in § 2 the general method of arriving at the generating function of the photocounts corresponding to a gaussian beam of arbitrary spectral profile with a fixed centre frequency. In § 3, we deal with the superposition of two incoherent beams. This is followed in § 4 by a complete analysis of the photocounts of beams characterized by a general complex autocorrelation function.

2. General theory of photocounts of an incoherent gaussian beam

At the outset, we observe that the individual photocounts constitute a stochastic point process on the time axis. However, by virtue of the mutual independence of the counting device as well as the intensity of the beam, it is eminently reasonable to characterize the counting process as a doubly stochastic Poisson process†. If $V(t)$ is the analytic signal the positive random variable characterizing the doubly stochastic process is given by

$$I(t) = V^*(t)V(t). \tag{2.1}$$

We shall assume that $V(t)$ is a stationary gaussian random process. Denoting the generating function governing the number of photocounts in any interval $(0, T)$ by $Q(s, T)$, we have

$$Q(s, T) = E\left\{\exp\left(-s\int_0^T I(t)\,\mathrm{d}t\right)\right\}. \tag{2.2}$$

† In fact, doubly stochastic Poisson processes were first introduced by Cox (1955) in connection with the study of the stops of a loom due to weft breaks.

To evaluate the expectation value implied by (2.2), we seek a Loéve expansion of the random variable $V(t)$ in terms of an orthogonal set of functions over the interval $(0, T)$ (see Loéve 1963 and Davenport and Root 1958)

$$V(t) = \sum_m C_m \phi_m(t) \tag{2.3}$$

where the random coefficients $\{C_m\}$ satisfy the relation

$$E\{C_m^* C_n\} = \lambda_m \delta_{mn} \tag{2.4}$$

and the ϕ_m are the eigenfunctions corresponding to the eigenvalues λ_m of the integral equation

$$\int_0^T \Gamma(t, t')\phi_m(t)\,\mathrm{d}t = \lambda_m \phi_m(t') \tag{2.5}$$

where

$$\Gamma(t, t') = E\{V(t)V^*(t')\} \tag{2.6}$$

is the unnormalized autocorrelation function characterizing the beam. We note that the random coefficients $\{C_m\}$ have a complex distribution

$$\text{Probability}(\{C_m\}) = \prod_m \frac{1}{\pi\lambda_m} \exp\left(-\frac{|C_m|^2}{\lambda_m}\right). \tag{2.7}$$

Using the representation (2.7), we can express the generating function of the photocounts in terms of the eigenvalues λ_m

$$Q(s, T) = \prod_k (1+s\lambda_k)^{-1}. \tag{2.8}$$

An explicit expression for the generating function has been obtained by Jakeman and Pike (1968) when Γ is given by

$$\Gamma(t, t') = \bar{I} \exp\{-\Gamma|t-t'| + \mathrm{i}\omega_0(t-t')\}. \tag{2.9}$$

We shall demonstrate in this paper the possibility of obtaining $Q(s, T)$ explicitly for any arbitrary Γ. In this section, we shall assume that Γ is of the form

$$\Gamma(t, t') = f(|t-t'|)\exp\{\mathrm{i}\omega_0(t-t')\} \tag{2.10}$$

where f is a real valued function defined on the positive real axis. In the final section, we shall remove the reality condition to describe the superposition of light beams of different centre-frequencies. Setting

$$\Phi_k(t) = \phi_k \exp(\mathrm{i}\omega_0 t) \tag{2.11}$$

we find that $\Phi_k(t)$ satisfies the equation†

$$\int_0^T f(|t-t'|)\Phi(t)\,\mathrm{d}t = \lambda\Phi(t'). \tag{2.12}$$

To solve the above integral equation, we introduce the Laplace transform of Φ by

$$\psi(p) = \int_0^T \Phi(t)\exp(-pt)\,\mathrm{d}t. \tag{2.13}$$

† From here on, we drop the suffix k for notational convenience.

We shall assume that f has a Laplace transform f^*. Substituting in equation (2.12), the inversion formula

$$f(|t-t'|) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} f^*(z) \exp(z|t-t'|) \mathrm{d}z \tag{2.14}$$

where σ is real and positive, we obtain

$$\frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} f^*(z) \, \mathrm{d}z \left(\frac{\psi(z) \exp\{(z-p)T\} - \psi(p)}{z-p} + \frac{\psi(-z) - \psi(p)}{z+p} \right) = \lambda \psi(p). \tag{2.15}$$

We note that $f^*(z)$ is analytic in the half plane $\mathrm{Re}\, z > 0$ by virtue of the bounded nature of the autocorrelation function as defined by (2.10). To make further progress, we shall assume that $f^*(z)$ is a rational function of z, that is

$$f^*(z) = \frac{g(z)}{h(z)} \tag{2.16}$$

where $h(z)$ is a polynomial of degree n and $g(z)$ a polynomial of degree not exceeding $(n-1)$. We evaluate the line integral occurring on the left hand side of equation (2.15) by choosing σ to be greater than $\max(0, \mathrm{Re}\, p, -\mathrm{Re}\, p)$. We observe that ψ is an entire function of its argument and using the definition we obtain the following estimate for large $\mathrm{Re}\, p$:

$$|\psi(p)| \simeq \frac{1 - \exp(-\mathrm{Re}\, pT)}{\mathrm{Re}\, p} \qquad |\mathrm{Re}\, p| \gg 1. \tag{2.17}$$

Using the estimate (2.17), we evaluate the line integral corresponding to the second and the fourth terms of the integrand by closing to the right and conclude that it is zero.

Next, we observe that the poles of $f^*(z)$ are due to the zeros of $h(z)$ which are necessarily located in the half plane $\mathrm{Re}\, z < 0$. Assuming that $h(z)$ has m distinct zeros at the points $z_1, z_2, \ldots, z_m$ with respective multiplicities $l_1, l_2, l_3, \ldots, l_m$ we evaluate the line integrals corresponding to the first and third terms on the left hand side of equation (2.15) by converting them into contour integrals. We thus obtain

$$\sum_{k=1}^{m} (a_k \exp(-pT) + b_k) + (f^*(p) + f^*(-p))\psi(p) = \lambda \psi(p) \tag{2.18}$$

where

$$a_k = \sum_{j=0}^{l_k - 1} \frac{F_{jk}}{(p - z_k)^{j+1}} \tag{2.19a}$$

$$F_{jk} = \frac{(-1)}{(l_k - i - j)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{l_k - i - j} (\mathrm{e}^{zT} \psi(z) f^*(z) (z - z_k)^{l_k}) \tag{2.19b}$$

$$b_k = \sum_{j=0}^{l_k - 1} \frac{G_{jk}}{(p + z_k)^{j+1}} \tag{2.20a}$$

$$G_{jk} = \frac{(-1)^j}{(l_k - i - j)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{l_k - i - j} (\psi(-z) f^*(z) (z - z_k)^{l_k}). \tag{2.20b}$$

Solving for $\psi(p)$ we obtain

$$\psi(p) = \frac{h(p)h(-p)\Sigma_{k=1}^{m}(a_k e^{-pT}+b_k)}{\lambda h(p)h(-p)-g(p)h(-p)-g(-p)h(p)}. \tag{2.21}$$

Observing that $\psi(p)$ is an entire function and that the denominator is a polynomial of degree $2n$, we note that the numerator should vanish precisely at the $2n$ zeros of the denominator. These conditions on the numerator yield a set of $2n$ equations for the $2n$ unknown constants F_{jk} and G_{jk} occurring in equations (2.19*b*) and (2.20*b*). These conditions result in the form

$$[D]\boldsymbol{K} = 0 \tag{2.22}$$

where $\boldsymbol{K}$ is a $2n$ dimensional vector whose components are given by

$$K_j = \begin{cases} F_{\alpha\beta} & \text{for } j \leqslant n \\ G_{\gamma\beta} & \text{for } j > n \end{cases} \tag{2.23a}$$

where

$$\alpha = j-1-L_\theta \qquad \beta = \theta+1 \qquad L_\theta < j \leqslant L_{\theta+1}$$

$$\theta = 0, 1, 2, \ldots, (m-1) \qquad j \leqslant n \tag{2.23b}$$

$$\gamma = j-n-1-L_\theta \qquad \beta = \theta+1 \qquad L_\theta < (j-n) \leqslant L_{\theta+1}$$

$$\theta = 0, 1, 2, \ldots, (m-1) \qquad j > n \tag{2.23c}$$

and

$$L_\theta = l_0+l_1+\ldots+l_\theta \qquad L_0 = l_0 = 0 \qquad L_m = n. \tag{2.23d}$$

The elements of $[D]$ are given by

$$\begin{aligned} D_{ij} &= \frac{h(p_i)h(-p_i)\exp(-p_iT)}{(p_i-z_{\theta+1})^{j-L_\theta}} \qquad L_\theta < j \leqslant L_{\theta+1} \qquad j \leqslant n \\ D_{ij} &= \frac{h(p_i)h(-p_i)}{(p_i+z_{\theta+1})^{j-L_\theta-n}} \qquad L_\theta < (j-n) \leqslant L_{\theta+1} \qquad j > n \end{aligned} \tag{2.24}$$

where $p_1, p_2, \ldots, p_{2n}$, the zeros of $\lambda h(p)h(-p)-g(p)h(-p)-g(-p)h(p)$ may be regarded as the $2n$ branches of a multiple-valued function of λ defined by

$$\lambda h(p)h(-p)-g(p)h(-p)-g(-p)h(p) = 0. \tag{2.25}$$

We notice that equation (2.22) represents a homogeneous system of equations for the components of $\boldsymbol{K}$ and in order that the solution be nontrivial, we have

$$|D| = 0. \tag{2.26}$$

Since λ is the only unknown parameter in the determinant, the above equation is the eigenvalue equation. Since p_i are functions of λ defined by equation (2.25), it is clear there are an infinite number of eigenvalues. If we denote $|D|$ by $F(1/\lambda)$, it is easy to see that F is not an entire function of its argument in view of its not returning to its original value if we go along any arbitrary closed contour containing the origin. However, it can easily be proved that F is an analytic function in any bounded domain of the cut ξ plane ($\xi = 1/\lambda$). Thus the nonanalytic property in the ξ plane arises essentially from the multiple-valued nature of the function defined by equation (2.25). However, it is easily seen that the determinant is an alternant with reference to the parameters $p_1, p_2, \ldots, p_{2n}$ and the different values of $|D(\xi)|$ corresponding to the different branches are obtained by the permutations of the parameters $p_1, p_2, \ldots, p_{2n}$. By the characteristic property of the determinant, all the values are the same except for the sign. Thus we can construct an entire function from $|D(\xi)|$ by multiplying or dividing it by an appropriate alternant in $p_1, p_2, \ldots, p_{2n}$. Defining $P(\xi)$ by

$$P(\xi) = \frac{|D(\xi)|}{|D_0(\xi)|} \tag{2.27}$$

where the elements of the determinant $|D_0(\xi)|$ are given by

$$|D_0(\xi)|_{ij} = p_i^{j-1} \qquad i, j = 1, 2, \ldots, 2n. \tag{2.28}$$

We note that $|D_0(\xi)|$ is a factor of $|D(\xi)|$ and zeros of $|D_0(\xi)|$ do not contribute to the eigenvalues of the original integral equation (2.12). Now $P(\xi)$ is an entire function and its zeros are directly related to the eigenvalues of the basic integral equation (2.12). We can obtain a representation for $P(\xi)$ by using the Hadamard–Weierstrass theorem relating to the canonical representation of an entire function (see, for example, Hille 1962). To do this, we first determine the order of $P(\xi)$. We notice that the order of $P(\xi)$ is related to the behaviour of p_i for large values of ξ. If the degree of $g(p)$ is m (which is always less than n), it follows from equation (2.25) that for large $|\xi|$:

$$|p| \simeq C|\xi|^{1/2(n-m)} \tag{2.29}$$

where C is a positive constant. Observing that the dependence of $P(\xi)$ on ξ, which, in turn, is related to the dependence of $|D(\xi)|$ on ξ, is dominated by terms of the form $\exp(-p_i T)$, we conclude that the order of the entire function $P(\xi)$ is at most $\frac{1}{2}$.

Thus we have the representation for $P(\xi)$

$$P(\xi) = P(0) \prod_k \left(1 - \frac{\xi}{\xi_k}\right) \tag{2.30}$$

where the constant $P(0)$ is determined using equation (2.27). Comparing the above representation with equation (2.8), we obtain the following explicit expression for the generating function

$$Q(s, T) = \frac{P(0)}{P(-s)}. \tag{2.31}$$

The results corresponding to the special case dealt with by Jakeman and Pike (1968) and Bédard (1966) can be easily deduced from (2.31). As an example we derive the generating function for the case of gaussian–lorentzian light. In this case, the autocorrelation function is given by

$$E\{V(t)V^*(t')\} = \bar{I} \exp\{-\Gamma|t-t'| + \mathrm{i}\omega_0(t-t')\} \tag{2.32}$$

where $\bar{I}$ is the average intensity, ω_0 the centre frequency and Γ the halfwidth at half height. The relevant integral equation is

$$\int_0^T \bar{I} \exp(-\Gamma|t-t'|)\Phi(t)\,\mathrm{d}t = \lambda\Phi(t'). \tag{2.33}$$

Taking Laplace transform of both sides of (2.33) and solving for $\psi(p)$, we get

$$\psi(p) = \frac{\bar{I}[(\Gamma-p)\exp\{-(\Gamma+p)T\}\psi(-\Gamma)+(\Gamma+p)\psi(\Gamma)]}{2\Gamma\bar{I}-\lambda(\Gamma^2-p^2)}. \tag{2.34}$$

To preserve the analyticity of $\psi(p)$, we now demand that the numerator of the right hand side of (2.34) should vanish at the two zeros p_1 and p_2 of the denominator. This yields

$$(\Gamma-p_1)\exp\{-(\Gamma+p_1)T\}\psi(-\Gamma)+(\Gamma+p_1)\psi(\Gamma) = 0 \tag{2.35a}$$

$$(\Gamma-p_2)\exp\{-(\Gamma+p_2)T\}\psi(-\Gamma)+(\Gamma+p_2)\psi(\Gamma) = 0. \tag{2.35b}$$

For nontrivial $\psi(-\Gamma)$ and $\psi(\Gamma)$, the determinant of the coefficients should vanish. This yields

$$D(\xi) = \begin{vmatrix} (\Gamma-p_1)\exp\{-(\Gamma+p_1)T\} & (\Gamma+p_1) \\ (\Gamma-p_2)\exp\{-(\Gamma+p_2)T\} & (\Gamma+p_2) \end{vmatrix} = 0. \tag{2.36}$$

We divide $D(\xi)$ by a suitable alternant in p_1 and p_2 in order to obtain an entire function $P(\xi)$. In this case, the alternant $D_0(\xi)$ is just p_2-p_1. Therefore, $P(\xi)$ is given by

$$P(\xi) = \frac{-1}{2(\Gamma^2-2\Gamma\bar{I}\xi)^{1/2}}\begin{vmatrix} (\Gamma-p_1)\exp\{-(\Gamma+p_1)T\} & (\Gamma+p_1) \\ (\Gamma-p_2)\exp\{-(\Gamma+p_2)T\} & (\Gamma+p_2) \end{vmatrix}. \tag{2.37}$$

Putting $\xi = 0$ in this, we get

$$P(0) = \frac{-1}{2\Gamma}\begin{vmatrix} 0 & 2\Gamma \\ 2\Gamma & 0 \end{vmatrix} = 2\Gamma \tag{2.38}$$

when $\xi = -s$

$$p_1 = -p_2 = \epsilon_s = (\Gamma^2+2\Gamma\bar{I}s)^{1/2}. \tag{2.39}$$

We therefore have

$$\begin{aligned} P(-s) &= \frac{-1}{2\epsilon_s}\begin{vmatrix} (\Gamma-\epsilon_s)\exp\{-(\Gamma+\epsilon_s)T\} & (\Gamma+\epsilon_s) \\ (\Gamma+\epsilon_s)\exp\{-(\Gamma-\epsilon_s)T\} & (\Gamma-\epsilon_s) \end{vmatrix} \\ &= 2\Gamma\,\mathrm{e}^{-\Gamma T}\left\{\cosh\epsilon_s T+\frac{1}{2}\left(\frac{\Gamma}{\epsilon_s}+\frac{\epsilon_s}{\Gamma}\right)\sinh\epsilon_s T\right\}. \end{aligned} \tag{2.40}$$

From (2.31), (2.38) and (2.40), we get

$$Q(s,T) = \mathrm{e}^{\Gamma T}\left\{\cosh\epsilon_s T+\frac{1}{2}\left(\frac{\epsilon_s}{\Gamma}+\frac{\Gamma}{\epsilon_s}\right)\sinh\epsilon_s T\right\}^{-1} \tag{2.41}$$

which agrees with the result arrived at by Jakeman and Pike (1968).

We note that the technique presented above yields the explicit solutions if $f^*(z)$ has a Mittag-Leffler expansion provided $f^*(z) \to 0$ at least as fast as z^{-1} for large values of $|z|$. In this case, we note that in the process of evaluation of the integral corresponding to the first and the third terms on the left hand side of (2.15), we can close the contour by a semicircular arc of radius R (R being so chosen that it does not pass through any of the poles). For any given $\epsilon > 0$ (ϵ being small), it is possible to make the modulus of the contribution of the integral along the circular arc less than ϵ by choosing R sufficiently large. Such a determination of R will give rise to (say) n poles within the contour so chosen. Without loss of generality, these can be chosen to be located at the points $z_1, z_2, \ldots, z_m$ with multiplicities $l_1, l_2, \ldots, l_m$. From this point onwards, the arguments that lead to the determination of the eigenvalues λ are applicable verbatim since this corresponds to an approximation of $f^*(z)$ by a rational function. Thus the only case that is left out is when $g^*(z)$ has a transcendental component. In this case, it is not clear how the eigenvalues could be determined.

3. Superposition of two incoherent lorentzian beams

We next proceed to discuss the photocounts of a mixture of two incoherent beams. This case is particularly interesting when the component beams are linearly independent and are completely polarized orthogonal with respect to each other, their effect being the same as that of partially polarized light, according to a well known result due to Mandel (1963). We observe that equations (2.1)–(2.8) are still valid provided we define $V(t)$ by

$$V(t) = V_1(t) + V_2(t) \tag{3.1}$$

where $V_1(t)$ and $V_2(t)$ are the analytic signals corresponding to the two superposed beams. For purposes of illustration, we assume that $V_1(t)$ and $V_2(t)$ are a pair of statistically independent gaussian processes and that their corresponding autocorrelation functions are given by

$$E\{V_1(t)V_1^*(t')\} = \bar{I}_1 \exp\{i\omega_0(t-t') - \Gamma|t-t'|\} \tag{3.2}$$

$$E\{V_2(t)V_2^*(t')\} = \bar{I}_2 \exp\{i\omega_q(t-t') - \Gamma'|t-t'|\}. \tag{3.3}$$

We assume that the second beam has its width equal to zero ($\Gamma' = 0$) for simplicity in purposes of illustration although no computational difficulty is experienced if $\Gamma' \neq 0$.

The basic integral equation in this case is given by

$$\int_0^T [\bar{I}_1 \exp\{i(\omega_0 - \omega_q)(t-t') - \Gamma|t-t'|\} + I_2]\Phi(t)\,dt = \lambda\Phi(t') \tag{3.4}$$

where

$$\Phi(t) = \phi(t)\exp(i\omega_q t). \tag{3.5}$$

Proceeding as before, we find that the Laplace transform $\psi(p)$ of $\Phi(t)$ is given by

$$\begin{aligned}\psi(p) = {} & (-\bar{I}_1\{i(\omega_0-\omega_q)-(\Gamma-p)\}\psi(-\Gamma-i(\omega_0-\omega_q))\\ & \times[\exp - T\{i(\omega_0-\omega_q)+(\Gamma+p)\}] + \{i(\omega_0-\omega_q)+(\Gamma+p)\}\\ & \times \bar{I}_1\psi(\Gamma - i(\omega_0-\omega_q)) + \bar{I}_2\{i(\omega_0-\omega_q)+(\Gamma+p)\}\\ & \times\{i(\omega_0-\omega_q)-(\Gamma-p)\}\{(1-e^{-pT})/p\}\psi(0))\\ & \times[\{i(\omega_0-\omega_q)-(\Gamma-p)\}\{i(\omega_0-\omega_q)+(\Gamma+p)\}+2\Gamma\bar{I}_1]^{-1}.\end{aligned} \tag{3.6}$$

We notice that the denominator of the right hand side of equation (3.6) has two zeros p_1 and p_2 given by

$$\Omega + p_1 T = (\gamma^2 - 2\gamma T\bar{I}_1\xi)^{1/2} \tag{3.7a}$$

$$\Omega + p_2 T = -(\gamma^2 - 2\gamma T\bar{I}_1\xi)^{1/2} \tag{3.7b}$$

where

$$\Omega = \mathrm{i}(\omega_0 - \omega_q)T \qquad \gamma = \Gamma T. \tag{3.7c}$$

As before, we insist that the numerator of the right hand side of equation (3.6) should vanish at p_1 and p_2. These conditions yield

$$\begin{aligned}
&-\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)-(\Gamma-p_1)\}\exp[-T\{\mathrm{i}(\omega_0-\omega_q)+(\Gamma+p_1)\}]\\
&\quad\times\psi(-\Gamma-\mathrm{i}(\omega_0-\omega_q))+\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)+(\Gamma+p_1)\}\psi(\Gamma-\mathrm{i}(\omega_0-\omega_q))\\
&\quad+\bar{I}_2\{\mathrm{i}(\omega_0-\omega_q)+(\Gamma+p_1)\}\{\mathrm{i}(\omega_0-\omega_q)-(\Gamma-p_1)\}\\
&\quad\times\frac{1-\mathrm{e}^{-p_1T}}{p_1}\psi(0)=0
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&-\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)-(\Gamma-p_2)\}\exp[-T\{\mathrm{i}(\omega_0-\omega_q)+\Gamma+p_2\}]\\
&\quad\times\psi(-\Gamma-\mathrm{i}(\omega_0-\omega_q))+\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)+(\Gamma+p_2)\}\psi(\Gamma-\mathrm{i}(\omega_0-\omega_q))\\
&\quad+\bar{I}_2\{\mathrm{i}(\omega_0-\omega_q)+(\Gamma+p_2)\}\{\mathrm{i}(\omega_0-\omega_q)-(\Gamma-p_2)\}\\
&\quad\times\frac{1-\mathrm{e}^{-p_2T}}{p_2}\psi(0)=0.
\end{aligned} \tag{3.9}$$

We next observe that there are three unknown constants corresponding to the values of ψ at the points 0, $(\Gamma-\mathrm{i}(\omega_0-\omega_q))$ and $(-\Gamma-\mathrm{i}(\omega_0-\omega_q))$. Apart from equations (3.8) and (3.9), a third equation can be obtained by evaluating equation (3.6) at $p = 0$

$$\begin{aligned}
&-\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)-\Gamma\}\exp[-T\{\mathrm{i}(\omega_0-\omega_q)+\Gamma\}]\psi(-\Gamma-\mathrm{i}(\omega_0-\omega_q))\\
&\quad+\bar{I}_1\{\mathrm{i}(\omega_0-\omega_q)+\Gamma\}\psi(\Gamma-\mathrm{i}(\omega_0-\omega_q))\\
&\quad+[(T\bar{I}_2-\lambda)\{\mathrm{i}(\omega_0-\omega_q)+\Gamma\}\{\mathrm{i}(\omega_0-\omega_q)-\Gamma\}-2\Gamma\bar{I}_1]\psi(0)=0.
\end{aligned} \tag{3.10}$$

Equations (3.8)–(3.10) are a set of homogeneous equations for the three constants $\psi(-\Gamma-\mathrm{i}(\omega_0-\omega_q))$, $\psi(\Gamma-\mathrm{i}(\omega_0-\omega_q))$ and $\psi(0)$. In order that the solution be nontrivial, we have

$$D(\xi) = -\ \bar{I}_1^2\exp(-\gamma)\times\begin{vmatrix}
(\epsilon-\Gamma)\,\mathrm{e}^{-\epsilon T} & (\epsilon+\Gamma) & I_2(\epsilon^2-\Gamma^2)(1-\mathrm{e}^{\Omega-\epsilon T})/(\epsilon-\Omega/T)\\
(-\epsilon-\Gamma)\,\mathrm{e}^{-\epsilon T} & (-\epsilon+\Gamma) & I_2(\epsilon^2-\Gamma^2)(1-\mathrm{e}^{\Omega+\epsilon T})/(-\epsilon-\Omega/T)\\
(\Omega/T-\Gamma)\xi\,\mathrm{e}^{-\Omega} & (\Omega/T+\Gamma)\xi & \{(\xi T\bar{I}_2-1)(\Omega^2/T^2-\Gamma^2)-2\Gamma\bar{I}_1\xi\}
\end{vmatrix} = 0 \tag{3.11}$$

where

$$\Omega+p_1T = (\gamma^2-2\gamma T\bar{I}_1\xi)^{1/2} = \epsilon T$$

and

$$\Omega+p_2T = -(\gamma^2-2\gamma T\bar{I}_1\xi)^{1/2} = -\epsilon T. \tag{3.12}$$

We note that the determinant $|D(\xi)|$ occurring in equation (3.11) is an alternating function of p_1 and p_2 and hence it is easy to construct an entire function $P(\xi)$ by dividing the determinant by $p_1 - p_2$

$$P(\xi) = \frac{1}{p_1 - p_2} |D(\xi)|$$

$$= \frac{\bar{I}_1^2 \mathrm{e}^{-\gamma}}{2\epsilon}$$

$$\times \begin{vmatrix} (\epsilon-\Gamma)\mathrm{e}^{-\epsilon T} & (\epsilon+\Gamma) & \bar{I}_2(\epsilon^2-\Gamma^2)(1-\mathrm{e}^{\Omega-\epsilon T})/(\epsilon-\Omega/T) \\ (-\epsilon-\Gamma)\mathrm{e}^{\epsilon T} & (-\epsilon+\Gamma) & \bar{I}_2(\epsilon^2-\Gamma^2)(1-\mathrm{e}^{\Omega+\epsilon T})/(-\epsilon-\Omega/T) \\ (\Omega/T-\Gamma)\xi\,\mathrm{e}^{-\Omega} & (\Omega/T+\Gamma)\xi & \{(\xi T\bar{I}_2-1)(\Omega^2/T^2-\Gamma^2)-2\Gamma\bar{I}^1\xi\} \end{vmatrix}. \tag{3.13}$$

The value $P(0)$ can be calculated from equations (3.11) and (3.13)

$$P(0) = 2\bar{I}_1^2\Gamma\left(\Gamma^2 - \frac{\Omega^2}{T^2}\right). \tag{3.14}$$

Thus, we have

$$G(s, T) = \frac{P(0)}{P(-s)}$$

$$= \left\{4\Gamma\epsilon_s\left(\Gamma^2-\frac{\Omega^2}{T^2}\right)\exp(\gamma)\right\}\left[4\Gamma\epsilon_s\left\{(sT I_2+1)\left(\Gamma^2-\frac{\Omega^2}{T^2}\right)\right.\right.$$

$$\left.+2\Gamma\bar{I}_1 s\right\}\left\{\cosh\epsilon_s T+\frac{1}{2}\left(\frac{\Gamma}{\epsilon_s}+\frac{\epsilon_s}{\Gamma}\right)\sinh\epsilon_s T\right\}$$

$$+\left(\frac{\Omega}{T}+\Gamma\right)s\bar{I}_2\left(\frac{\epsilon_s+\Gamma}{\epsilon_s-(\Omega/T)}(\epsilon_s^2-\Gamma^2)\{1-\exp(\Omega-\epsilon_s T)\}\,\mathrm{e}^{\epsilon_s T}\right.$$

$$\left.-\frac{\epsilon_s-\Gamma}{\epsilon_s+(\Omega/T)}(\epsilon_s^2-\Gamma^2)\{1-\exp(\Omega+\epsilon_s T)\}\,\mathrm{e}^{-\epsilon_s T}\right)$$

$$-\left(\frac{\Omega}{T}-\Gamma\right)s\bar{I}_2\,\mathrm{e}^{-\Omega}\left(-\frac{\epsilon_s+\Gamma}{\epsilon_s+(\Omega/T)}(\epsilon_s^2-\Gamma^2)\{1-\exp(\Omega+\epsilon_s T)\}\right.$$

$$\left.\left.+\frac{\epsilon_s-\Gamma}{\epsilon_s-(\Omega/T)}(\epsilon_s^2-\Gamma^2)\{1-\exp(\Omega-\epsilon_s T)\}\right)\right]^{-1} \tag{3.15}$$

where

$$\epsilon_s = (\Gamma^2+2\Gamma\bar{I}_1 s)^{1/2}.$$

We can now consider two special cases:

(i) $\Omega = 0$. $\Gamma \neq 0$. This corresponds to two beams with identical centre frequencies. In this case, the generating function is given by

$$G(s, T) = 4\Gamma^3\epsilon_s\,\mathrm{e}^{\gamma}\left[4\Gamma\epsilon_s(\Gamma^2(s\bar{I}_2 T+1)+2\Gamma\bar{I}_1 s)\right.$$

$$\times\left\{\cosh\epsilon_s T+\frac{1}{2}\left(\frac{\Gamma}{\epsilon_s}+\frac{\epsilon_s}{\Gamma}\right)\sinh\epsilon_s T-2\Gamma s\bar{I}_2\right\}(\epsilon_s^2-\Gamma^2)$$

$$\left.\times\left\{\left(\frac{\epsilon_s+\Gamma}{\epsilon_s}\right)(1-\mathrm{e}^{\epsilon_s T})-\left(\frac{\epsilon_s-\Gamma}{\epsilon_s}\right)(1-\mathrm{e}^{-\epsilon_s T})\right\}\right]^{-1}. \tag{3.16}$$

(ii) $\Omega = 0, \Gamma = 0$. This corresponds to the case of both the beams having zero width centred about the same frequency. In this case, the generating function is

$$G(s, T) = \{(s\bar{I}_1 T+1)(s\bar{I}_2 T+1)\}^{-1}. \tag{3.17}$$

The procedure discussed above needs modification if one of the components of the mixture is a coherent beam. Since a coherent beam is nothing but a harmonic signal with a random phase the mixture can no longer be described by a complex field whose real and imaginary parts represent a pair of gaussian random processes. In such a case, we seek an expansion of the amplitude corresponding to the mixture on the lines followed by Jakeman and Pike (1969) and the expression for the generating function turns out to be a little more complicated than equation (2.6) by a factor depending on the explicit form of $\psi(p)$. However, this does not offer any computational difficulty as long as the factor has a Laurent expansion about the point 1. The explicit method of calculation for the general case will be discussed in a separate paper.

4. Photocounts of beams with complex autocorrelation functions

We now proceed to consider the superposition of gaussian beams of arbitrary bandwidth as well as centre frequencies. For the purpose of this discussion, equations (2.1)–(2.8) are still valid. The autocorrelation function Γ shall be assumed to be of the form

$$\Gamma(t, t') = f_1(t-t')+f_2(t-t') \tag{4.1}$$

where f_1 and f_2 are real valued, even and odd functions respectively. These conditions ensure the hermitean nature of the autocorrelation function. We wish to solve the integral equation (2.5), using the explicit form of $\Gamma(t, t')$ given by equation (4.1).

Defining f_1^* and f_2^* as the Laplace transforms of f_1 and f_2 and substituting the inversion formulae for f_1 and f_2 into equation (2.5), we obtain the following equation for the Laplace transform $\hat{\psi}$ of ϕ:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}((f_1^*(z)+if_2^*(z))[\hat{\psi}(z)\exp\{(z-p)T\}-\hat{\psi}(p)](z-p)^{-1}$$
$$+(f_1^*(z)-if_2^*(z))(\hat{\psi}(-z)-\hat{\psi}(p))(z+p)^{-1})\,dz = \lambda\hat{\psi}(p). \tag{4.2}$$

We again assume that f_1^* and f_2^* are rational functions of z

$$f_1^*(z) = \frac{h_1(z)}{h_2(z)} \tag{4.3a}$$

$$f_2^*(z) = \frac{h_3(z)}{h_4(z)}. \tag{4.3b}$$

h_1, h_2, h_3 and h_4 are polynomials of z, h_2 of degree m and h_1 of degree not greater than $(m-1)$, h_4 of degree n and h_3 of degree not exceeding $(n-1)$. As before, the line integrals involving $\hat{\psi}(p)$ in the integrand are seen to vanish. The poles of f_1^* and f_2^* are due respectively to the zeros of $h_2(z)$ and $h_4(z)$ which again are located in the half plane $\mathrm{Re}\, z < 0$. We assume that $h_2(z)$ has q distinct zeros at the points $z_1, z_2, \ldots, z_q$ with multiplicities $l_1, l_2, \ldots, l_q$ and $h_4(z)$ has r distinct zeros at the points $z'_1, z'_2, \ldots, z'_r$ with multiplicities $l'_1, l'_2, \ldots, l_r$. We proceed as before to evaluate the line integrals corresponding to the terms $(f_1^*(z)+\mathrm{i}f_2^*(z))\hat{\psi}(z)\exp\{(z-p)T\}/(z-p)$ and $(f_1^*(z)-\mathrm{i}f_2^*(z)) \times \hat{\psi}(-z)/(z+p)$ by converting them into contour integrals. We obtain

$$\hat{\psi}(p) = \left\{h_2(p)h_2(-p)h_4(p)h_4(-p)\left(\sum_{k=1}^{q}(a_k\,\mathrm{e}^{-pT}+b_k)+\sum_{k=1}^{q}(a'_k\,\mathrm{e}^{-pT}+b'_k)\right)\right\}$$
$$\times\{\lambda h_2(p)h_2(-p)h_4(p)h_4(-p)-h_1(p)h_2(-p)h_4(p)h_4(-p)$$
$$-h_1(-p)h_2(p)h_4(p)h_4(-p)-\mathrm{i}h_3(p)h_2(p)h_2(-p)h_4(-p)$$
$$+\mathrm{i}h_3(-p)h_2(p)h_2(-p)h_4(p)\}^{-1} \tag{4.4}$$

where a_k, b_k, a'_k and b'_k are given by

$$a_k = \sum_{j=0}^{l_k-1}\frac{F_{jk}}{(p-z_k)^{j+1}} \tag{4.5a}$$

$$F_{jk} = \frac{-1}{(l_k-i-j)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{l_k-i-j}(\mathrm{e}^{zT}\hat{\psi}(z)f_1^*(z)\,(z-z_k)^{l_k}) \tag{4.5b}$$

$$b_k = \sum_{j=0}^{l_k-1}\frac{G_{jk}}{(p+z_k)^{j+1}} \tag{4.5c}$$

$$G_{jk} = \frac{(-1)^j}{(l_k-1-j)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{l_k-1-j}(\hat{\psi}(-z)f_1^*(z)(z-z_k)^{l_k}) \tag{4.5d}$$

$$a'_k = \sum_{j=0}^{l'_k-1}\frac{F'_{jk}}{(p-z'_k)^{j+1}} \tag{4.5e}$$

$$F'_{jk} = \frac{(-1)}{(l'_k-1-j)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{l'_k-1-j}(\mathrm{i}f_2^*(z)\,\mathrm{e}^{zT}\hat{\psi}(z)(z-z'_k)^{l'_k}) \tag{4.5f}$$

$$b'_k = \sum_{j=0}^{l'_k-1}\frac{G'_{jk}}{(p+z_k)^{j+1}} \tag{4.5g}$$

$$G'_{jk} = \frac{(-1)^{j+1}}{(l'_k-1-j)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{l'_k-1-j}(\mathrm{i}f_2^*(z)\hat{\psi}(-z)(z-z'_k)^{l'_k}). \tag{4.5h}$$

Again observing that $\hat{\psi}(p)$ is an entire function and that the denominator on the right hand side of equation (4.4) is a polynomial of degree $2(m+n)$, we note that the numerator on the right hand side of (4.4) should vanish precisely at the $2(m+n)$ zeros of the denominator. These conditions on the numerator yield a set of $2(m+n)$ equations for the $2(m+n)$ unknown constants, F_{jk}, G_{jk}, F'_{jk} and G'_{jk} occurring in equations (4.5). These result in the form

$$[D]\boldsymbol{K}' = 0 \tag{4.6}$$

where $\boldsymbol{K}'$ is a $2(m+n)$ dimensional vector whose elements are defined in a manner quite analogous to equation (2.23) whilst the elements of $[D]$ are expressed in the same manner as equation (2.24). From this point onwards, the steps run exactly parallel to those of § 2 and it is possible to construct an entire function $P(\xi)$ whose representation is given by equation (2.30) and which is connected to the generating function by equation (2.31).

In special cases, it may turn out that the steps are even simpler than what appears from the general method described above. For instance, the discussion in § 3 relating to the mixing of two incoherent lorentzian beams is a special case where

$$f_1(t-t') = \bar{I}_1 \exp(-\Gamma|t-t'|) \cos \omega_0(t-t') + \bar{I}_2 \cos \omega_q(t-t') \tag{4.7a}$$

$$f_2(t-t') = \bar{I}_1 \exp(-\Gamma|t-t'|) \sin \omega_0(t-t') + \bar{I}_2 \sin \omega_q(t-t'). \tag{4.7b}$$

As we have seen, this only gives rise to a 3×3 determinant and the steps leading to the solution are an obvious modification of the method presented in § 2.

In conclusion, we wish to observe that the photocount statistics of partially polarized beams can be regarded as having been explicitly solved since, as said earlier, a partially polarized beam can be regarded as equivalent to the superposition of two incoherent beams completely polarized orthogonally to each other, and linearly independent of each other. It is also worthwhile to note that the same method is applicable for the determination of the generating function of the n fold counting statistics.

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